The Second Painleve Transcendent And Nonlinear Optical Propagation

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Abstract: A new transformation for the complex amplitude of propagating electric fields in optical materials shows a family of exact non-stationary solutions to the 1+1 and 1+2 dimensional nonlinear Schrodinger equations. These solutions take the form of a second Painleve transcendent. The propagation of electric fields in materials, such as optical fibers, is described in terms of the nonlinear Schrodinger equation where the nonlinear term is proportional to the intensity of the field. The solutions given here for the 1+1 dimensional equation propagate in the dispersive medium with no change in their shape or amplitude but with a curved path. In addition, the Painleve solution is extended to the 1+2 dimensional equation, giving a new two-dimensional cosine tapered structure that propagates without changing shape or amplitude.

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The propagation of electric fields in materials, such as optical fibers, is described in terms of the nonlinear Schrodinger equation where the nonlinear term is proportional to the intensity of the field. A set of solutions in the form of a second Painleve transcendent describe propagate in dispersive media with no change in their shape or amplitude. Part A of this paper develops the one-dimensional propagation in time. This part is a reformulation, with additional results, of the work previously published (J.A. Giannini, R.I. Joseph, "The Role of the Second Painleve Transcendent in Nonlinear Optics," Phys. Lett. A, 141, No. 8-9, (1989) 417). Part B of this paper extends the results of part A to two-dimensional propagation in time. Both parts A and B represent work published in a Doctoral dissertation (J.A. Giannini, "The Propagation of Bright and Dark, Spatial and Temporal Solitons in Nonlinear Optical Materials," Doctoral Dissertation, Johns Hopkins University, (1991)).

A) Propagation In One Dimension — The Second Painleve Transcendent

The propagation of optical pulses in single mode lossless dispersive fibers is described by the nonlinear Schrodinger equation:

(a-1) $$iq_\xi + \tfrac{1}{2}Dq_{\tau\tau} + |q|^2q = 0.$$

The $q(\xi,\tau)$ is the complex amplitude of the electric field and the subscripts on "q" indicate partial derivatives with respect to the indicated variables. The D indicates the sign of the dispersion where D = –1 corresponds to normal group velocity dispersion and leads to a dark pulse, and D = +1 corresponds to anomalous dispersion, representing a bright pulse. The recent development of techniques for shaping short pulses [1-3] has generated renewed interest in the theory of dark solitons and their development has been studied numerically [4].

This paper presents a transformation of the nonlinear Schrodinger equation resulting in a family of analytic non-stationary solutions for pulse propagation which are related to the second Painleve transcendent. The solutions are applicable to both normal and anomalous dispersion but interest here is in the normal dispersion case (dark solitons).

The amplitude "q" of the electric field, modeled by the nonlinear Schrodinger equation, can be written in the form:

$$\text{(a-2)} \qquad q(\xi,\tau) = \sqrt{1/2}\; R(\xi,t)\, e^{i\theta(z,t)}.$$

The $t = 2\tau$, the $R(\xi, t)$ is the amplitude of the complex number, the $\xi \to -\xi$, and the $\theta(\xi, t)$ is the phase. For the stationary case leading to the well known hyperbolic secant and hyperbolic tangent soliton solutions [5-6], transformation to a reference frame fixed with respect to the moving wave uses the coordinate $\zeta = t - \xi/v$ where "v" is the speed of the waves. The non-stationary solutions derived here are obtained by the using the transformation

$$\text{(a-3)} \qquad \eta = t + \xi\cdot(f + g\xi)$$

This transforms the solution to a reference frame that is fixed with respect to an accelerating wave. The form for the phase $\theta(\xi,\tau)$ is chosen to be

$$\text{(a-4)} \qquad \theta = a\xi + b\tau + c\xi\tau + d\xi^2 + e\xi^3.$$

The "a", "b", "c", "d", "e", "f" and "g" are free parameters. Combining (a-1)–(a-4) and choosing parameters such that $b = \mp f/2$, $c = \mp g$, $d = \mp f\cdot g$, and $e = \mp 2g^2/3$ results in the following equation for R:

$$\text{(a-5)} \qquad \pm R'' = R(a \pm f^2/4 \mp g\eta) + 2R^3.$$

The prime denotes differentiation with respect to η. Note that if g = 0, (a-3) and (a-5) yield the usual stationary solutions to the nonlinear Schrodinger equation.

The final transformation of (a-5) uses

(a-6) $$x = (\eta - s)/p \quad \text{and} \quad y = R(\eta)/h.$$

Here, $s = (f^2/4 + a)/g$, $p^3 = -1/g$, and $h^2 = g^2/3$. This gives a family of solutions for "y" (and thus for $R(\eta)$) in the form of the Painleve transcendent of the second type,

(a-7) $$y'' = xy + 2y^3.$$

The double prime denotes a second differentiation with respect to "x". In this form, the positive sign on the cubic term corresponds to normal (positive) group velocity dispersion.

This characterization of the solutions to the nonlinear Schrodinger equation, although addressed in the hydrodynamic context, has not been associated previously with optical pulse propagation. Smith [7], who modeled large amplitude deep water waves, examined the behavior of the general asymptotic character of the solutions to an equation similar to Eq. (a-1).

$$ia_t = a_{\rho\rho} - \rho a + \beta|a|^2 a.$$

Although his results were obtained for a steady state, it is easy to show that by writing his amplitude as $a = B\, e^{i\theta}$, and using the transformations

$$\eta = \rho + t^2 \quad \text{and} \quad \theta = t \cdot (\rho + t^2/3).$$

This equation can be written in the form

$$iB_t = B_{\eta\eta} + \beta|B|^2 B.$$

Thus, solutions in terms of Painleve transcendent can be obtained without the steady state restriction.

Ablowitz [8] recognized this same behavior in his analysis of solutions to the Korteweq–de Vries equation, and, Miles [9] and Rosales [10] examined the Painleve Equation, (a-7), noting that in the asymptotic regime

$$y(x) = \alpha \mathrm{Ai}(x),$$

as $x \to \infty$, where Ai(x) is the Airy function of the first kind. This provides the initial conditions for "y" and y', in the numerical integration of (a-7) by using the corresponding values of the Airy function at a sufficiently large x(x = 10). (For the case of normal group velocity dispersion, the function y corresponds to what Miles and Rosales call F_-). For dark solitons, these solutions are bounded over all x when $0 < \alpha < 1$, decaying monotonically to zero in the positive x-direction. In the negative direction, the solutions oscillate as they decay. As $\alpha \to 0$, y/α approaches the Airy function for all "x". As $\alpha \to -1$, y/α becomes unbounded as $x \to -\infty$. Just as for the case of the Korteweg–de Vries equation [9], the parameter α is a measure of the relative importance of the nonlinearity and the dispersion in the system.

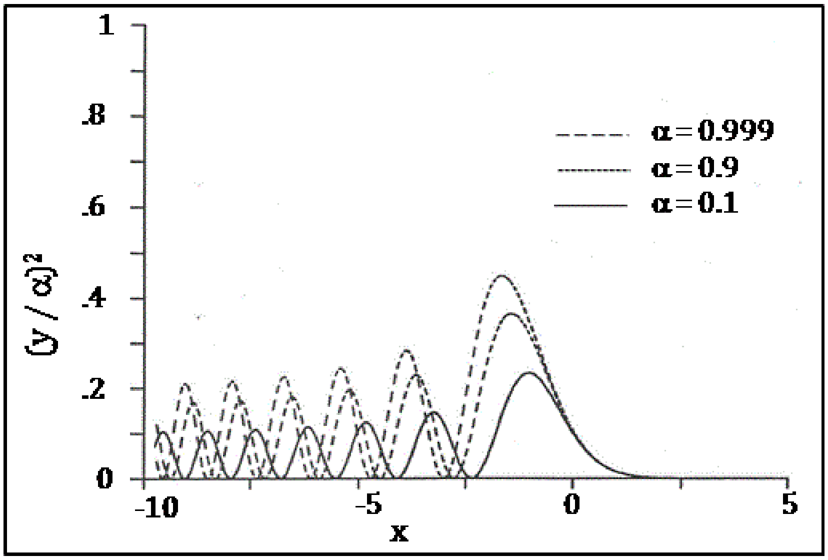

Fig. a-1 Integrated values of y/α for normal dispersion for three values of α. As $\alpha \to 0$, y/α approaches the Airy function. As $\alpha \to 1$, y/α becomes inbounded at $x = -\infty$

For the normal dispersion case, the square of the integrated values of y/α (proportional to the intensity) is shown for several values of α (α = 0.1, 0.9 and 0.999) (Fig. a-1). For the case of anomalous group velocity dispersion, corresponding to Miles and Rosales function F+, the solutions are bounded for all values of α (Fig. a-2).

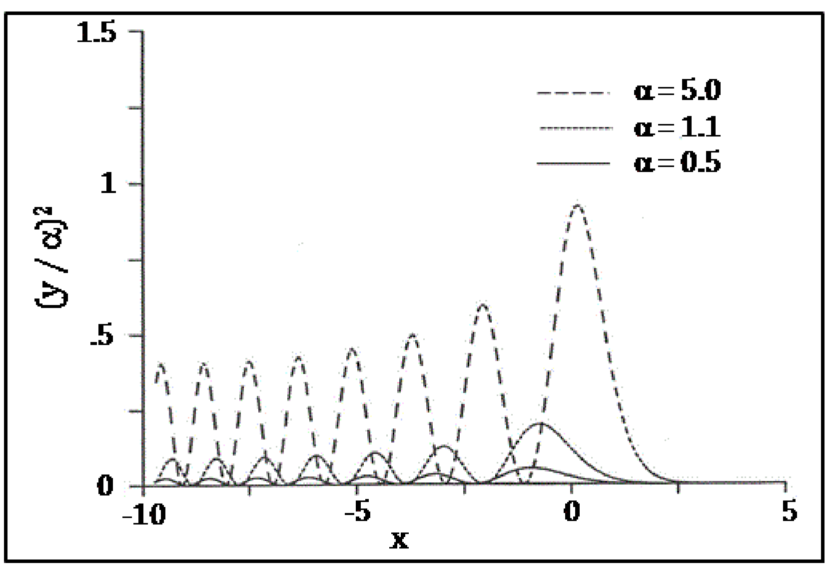

Fig. a-2 Integrated values of y/α for anomalous dispersion for three values of α. As $\alpha \to 0$, y/α approaches the Airy function.

B) Propagation in two spatial dimensions — a two-dimensional generalization of the second Painleve transcendent

The propagation of optical beams in dispersive materials with a Kerr nonlinearity can be described by the three-dimensional nonlinear Schrodinger equation:

(b-1) $$iq_z = \nabla_\perp^2 q + D|q|^2 q.$$

The q(x, y, z) is the complex amplitude of the electric field. The subscript "z" indicates partial differentiation with respect to the propagation direction and $\nabla_\perp^2$ is the Laplacian in the two transverse spatial directions. As before, the parameter D takes the values +1 for anomalous dispersion and –1 for normal dispersion. In (b-1), the Laplacian term, describing the dispersive effects of diffraction, counteracts the effect of the nonlinearity resulting in self-focusing or self-defocusing of the optical beam. This section presents a transformation for the three-dimensional nonlinear Schrodinger equation as given in (b-1) resulting in a family of non-stationary solutions related to the second Painleve transcendent. The solutions are applicable to both the normal and anomalous dispersion.

The amplitude of the electric field q(x,y,z) can be written in the form:

(b-2) $$q(x,y,z) = R(x, y, z)\, e^{i\theta(x,y,z)}.$$

Here, R(x,y,z) is the amplitude of a complex number and θ(x,y,z) is the phase. To solve (b-1) in terms of Painleve transcendents, the solution derived here is obtained by using the transformation:

(b-3a, b) $$t = x + g \cdot z^2 \quad \text{and} \quad s = y + h \cdot z^2.$$

The "g" and "h" are constants to be determined. Note that unlike the previous case, the independence of the "x" and "y" coordinates is maintained in the transformation to "t" and "s". This transforms the solution to a reference frame that is fixed with respect to an accelerating wave. The trajectory of any point on the wave is curving out from the x–y axes in both the "x" and "y" directions as the wave propagates. The form of the phase θ(x,y,z) is chosen to be:

(b-4) $$\theta(x,y,z) = a \cdot z \cdot x + b \cdot z \cdot y + c \cdot z^3.$$

The "a", "b" and "c" are parameters to be determined. Combining (b-1)–(b-2), (4-3)–(4-4) where z –> –z, and choosing parameters such that $g = -h$ and $c = -\frac{4}{3} a^2 D$, results in the following equation for R:

(b-5) $$D(R_{tt} + R_{ss}) = R\,(a \cdot t + b \cdot s) - R^3.$$

Transforming (b-5) using $t = p \cdot u$, $s = p' \cdot v$, and $R = k \cdot W$, where $p = p'$, $a = b$, and $k^2 = 2(D^2a^2)^{1/3}$, yields:

(b-6) $$W_{uu} + W_{vv} = W \cdot u + W \cdot v - 2DW^3.$$

Finally, rotating to a coordinate system at a 45 degree angle to the original x-y system: $\underline{u} = u + v$ and $\underline{v} = u - v$, and, letting $q = h \cdot \underline{u}$, $r = h \cdot \underline{v}$ and $V = b \cdot W$ with $h = 1/2^{1/3}$ and $b = 1/\sqrt{2^{1/3}}$, yields:

(b-7) $$V_{qq} + V_{rr} - V_q + 2D\,V^3 = 0.$$

This must be solved numerically. The boundary conditions used in the integration are determined by examining the form of (b-7) in the limit as $q \to \infty$. In this limit, the cubic term becomes negligible compared to the linear term and (b-7) reduces to

$$V_{qq}. + V_{rr} - V_q = 0.$$

This is solved exactly using separation of variables to give

$$V(r,q \to \infty) = \beta \cdot Ai(q+c) \cdot \cos(r\sqrt{c}).$$

Here β and "c" are free parameters and Ai is the airy function.

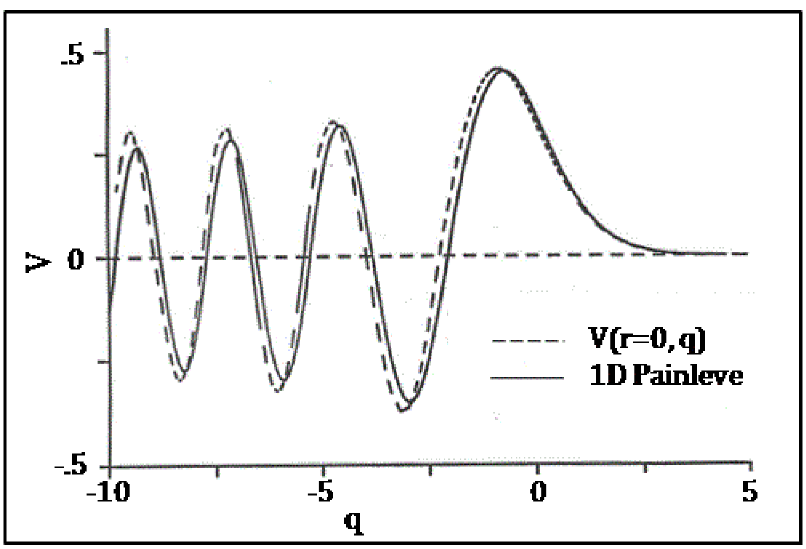

Fig. b-1 Solution to the two-dimensional extension of the second Painleve transcendent for anomalous dispersion with $c = \pi/20$ and $\beta = 1$. The view is along $r = 0$ (solid line) and is compared with the one-dimensional Painleve (dashed line)

The solution to (b-7) takes the form of a shifted second Painleve transcendent along the q-direction and a slightly perturbed cosine taper along the r-direction. The nulls of the cosine remain at the same "r" for all "q". In the limit c = 0, the solution reduces to the one-dimensional Painleve solution of part A (above) along the q-direction with a uniform amplitude in the r-direction. Like the one-dimensional solution, the two-dimensional solution is bounded for all values of β with anomalous dispersion. However, bounded solutions for normal dispersion only exist for $\beta < 1$. Fig. b-l shows the solution along the curve r = 0 with $c = \pi/20$ compared to the one-dimensional Painleve. Fig. b-2 shows the taper in the r-direction at the q-value for the function maximum compared with an unperturbed cosine, and Fig. b-3 shows the percent difference between the taper view and the cosine in Fig. b-2.

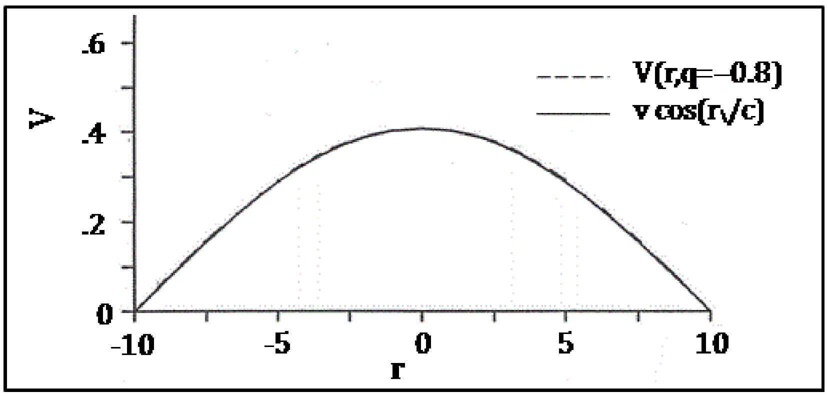

Fig. b-2 Solution to the two-dimensional extension of the second Painleve Transcendent for anomalous dispersion with $c = \pi/20$ and $\beta = 1$. The view is along $q = -0.8$ (solid line) and is compared with an exact cosine (dashed line).

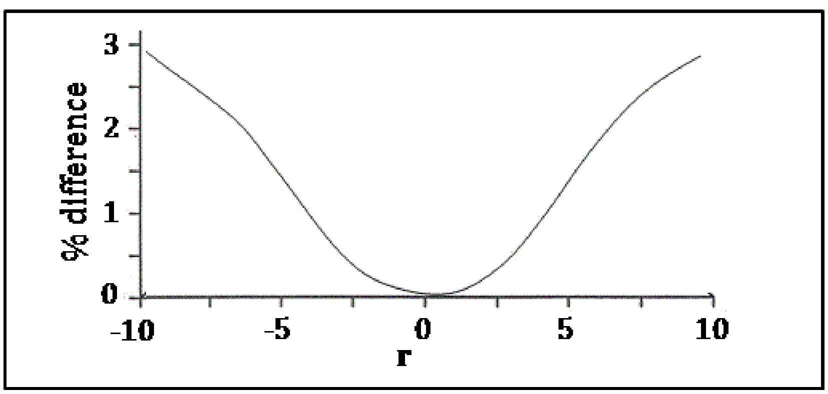

Fig. b-3 The percent difference between the two curves in fig. b.2 (the view along $q = -0.8$ and the exact cosine) where the maximum of V is v. The % = 100 x |V − v cos(r/c) | / v cos(r/c).

CONCLUSIONS

The results of this work can be summarized as follows. Nonlinear propagation is studied for optical materials where the third-order susceptibility $\chi^{(3)}$ is a significant contributor to the polarization. The effect resulting from $\chi^{(3)}$ that is examined is nonlinear refraction which is modeled in terms of the nonlinear Schrodinger equation. Solutions in terms of known functions and simple numerical integrals rather than numerical propagation simulations are obtained. A transformation for the nonlinear Schrodinger in one space-dimensional equation provides stable solutions that are described in terms of the second Painleve transcendent that is valid for both normal and anomalous dispersion for lossless propagation. For anomalous dispersion, bounded solutions exist for all values of the input while for normal dispersion there is a maximum value allowed for the initial amplitude. In addition to one-dimensional propagation, a transformation is presented for the nonlinear Schrodinger equation in three space-dimensions. A transformation similar the one used in the one-dimensional Painleve solution results in a two-dimensional structure that propagates with a curved trajectory. The solution is a full two-dimensional structure that is a shifted Painleve transcendent of the second kind in one dimension and a cosine taper in the second. For the case where the argument of the cosine is zero, the solution reduces exactly to the one dimensional Painleve solution previously mentioned.

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